How to use a natural conjugate distribution - an example

Behind the scenes

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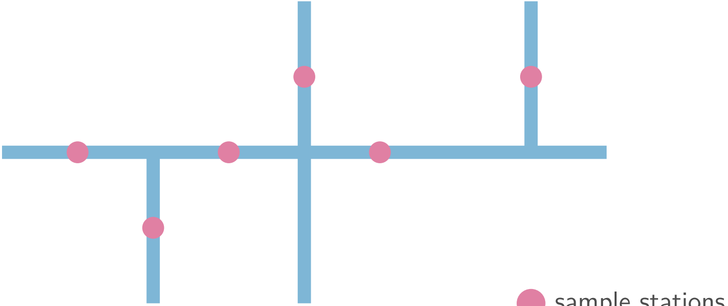
Coliform bacteria in the water supply network



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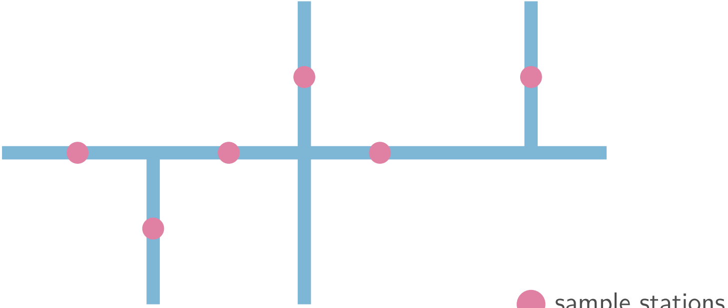

- Should we add chlorine in the water supply network to decrease the concentration of coliform bacteria?
- Example picked up in: Parent, E., & Bernier, J. (2007). *Le raisonnement bayésien: Modélisation et inférence*. Berlin: Springer e-books.

Building the model

- At each sampling stations:
  - 1 if coliform bacteria are detected,
  - 0 if not

So... let's define a set of random variables **i.i.d.** $X_i = \mathcal{B}(1, \theta)$ where θ is the **probability of detecting bacteria**.

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So... let's define a set of random variables **i.i.d.** $X_i = \mathcal{B}(1, \theta)$ where θ is the **probability of detecting bacteria**.

$$S = \sum_{i=1}^{n} X_i \quad (1)$$

$$\mathbb{P}(S = s) = \binom{n}{s} \theta^s (1 - \theta)^{n-s} \quad (2)$$

Prior information

Using a Bayesian framework allows us to combine different source of information:

- Information collected in previous years at the same period
- Similar network with similar risk

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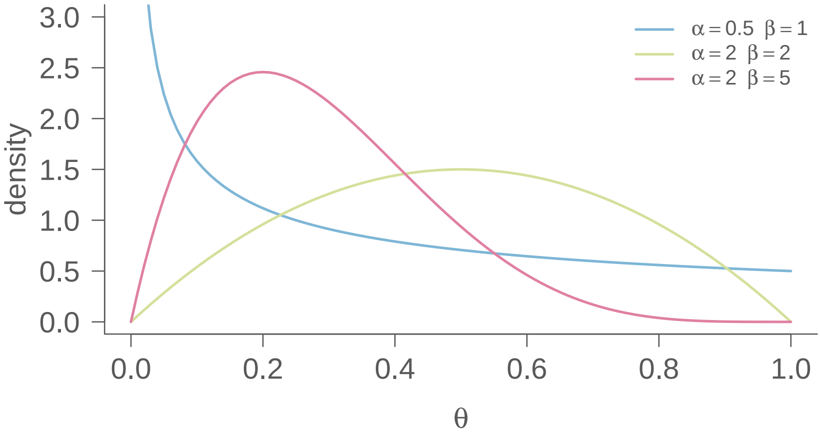
Let's use a prior distribution for θ, again we are modelers and we make assumptions that sound reasonable:

- for θ, the Beta distribution is well appropriate!

Beta distribution - dbeta



$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Beta distribution

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where:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \exp(-t)dt$$

$$\Gamma(z+1) = z\Gamma(z) \qquad k \in \mathcal{N} \quad \Gamma(k+1) = k!$$

NB: The Γ function is not the Gamma distribution

Beta distribution

$$\mathbb{E}(\theta) = \frac{\alpha}{\alpha + \beta}$$

$$\mathbb{V}(\theta) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Let's summarize

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$$\left[\theta|s\right] = \frac{\left[s|\theta\right]\left[\theta\right]}{\int_0^1 \left[s|\theta\right]\left[\theta\right]d\theta} \quad (5)$$

Demonstration

$$[\theta|s] = \frac{\binom{n}{s}\theta^s(1-\theta)^{n-s}\,[\theta]}{\int_0^1 \binom{n}{s}\theta^s(1-\theta)^{n-s}\,[\theta]\,d\theta} \tag{6}$$

$$[\theta|s] = \frac{\binom{n}{s}\theta^s(1-\theta)^{n-s}\,[\theta]}{\binom{n}{s}\int_0^1 \theta^s(1-\theta)^{n-s}\,[\theta]\,d\theta} \tag{7}$$

$$[\theta|s] = \frac{\theta^s(1-\theta)^{n-s}\,[\theta]}{\int_0^1 \theta^s(1-\theta)^{n-s}\,[\theta]\,d\theta} \tag{8}$$

$$[\theta|s] = \frac{\theta^s(1-\theta)^{n-s}\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\mathbb{1}_{[0,1]}}{\int_0^1 \theta^s(1-\theta)^{n-s}\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\mathbb{1}_{[0,1]}d\theta} \tag{9}$$

Demonstration

$$[\theta|s] = \frac{\frac{1}{B(\alpha,\beta)}\theta^s(1-\theta)^{n-s}\theta^{\alpha-1}(1-\theta)^{\beta-1}\mathbb{1}_{[0,1]}}{\frac{1}{B(\alpha,\beta)}\int_0^1 \theta^s(1-\theta)^{n-s}\theta^{\alpha-1}(1-\theta)^{\beta-1}\mathbb{1}_{[0,1]}d\theta} \quad (10)$$

$$[\theta|s] = \frac{\theta^s(1-\theta)^{n-s}\theta^{\alpha-1}(1-\theta)^{\beta-1}\mathbb{1}_{[0,1]}}{\int_0^1 \theta^s(1-\theta)^{n-s}\theta^{\alpha-1}(1-\theta)^{\beta-1}\mathbb{1}_{[0,1]}d\theta} \quad (11)$$

$$[\theta|s] = \frac{\theta^{s+\alpha-1}(1-\theta)^{n-s+\beta-1}\mathbb{1}_{[0,1]}}{\int_0^1 \theta^{s+\alpha-1}(1-\theta)^{n-s+\beta-1}\mathbb{1}_{[0,1]}d\theta} \quad (12)$$

$$
\begin{aligned}
[\theta|s] &= \frac{\theta^{\alpha+s-1}(1-\theta)^{n-s+\beta-1}\mathbb{1}_{[0,1]}}{B(n+\alpha-s, s+\beta)} && (13)\\
[\theta|s] &\sim \mathcal{B}eta(\alpha+s, n-s+\beta) && (14)
\end{aligned}
$$

So. . .

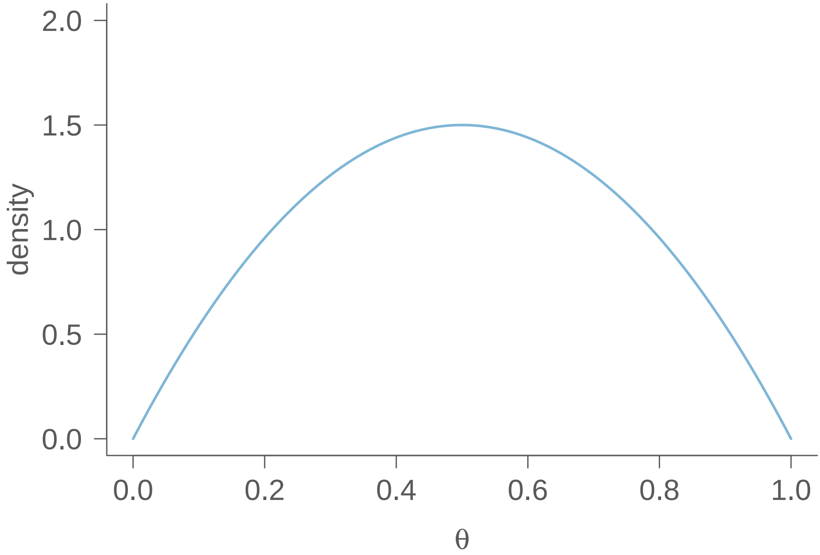
Figure 1: Fear the power of Math!

See: https://en.wikipedia.org/wiki/Conjugate_prior

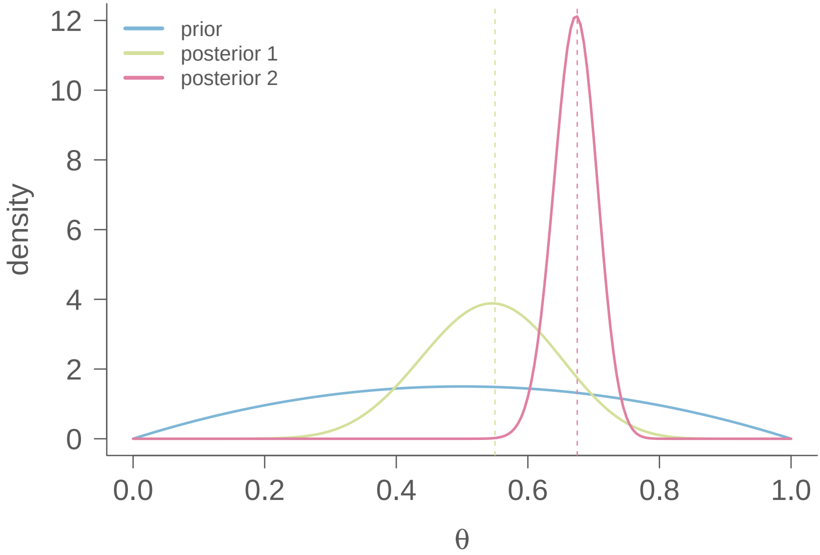
Example

```r
# simulated data
mydata1 <- rbinom(20, 1, .6)
mydata2 <- rbinom(200, 1, .6)
##
mydata3 <- rbinom(20, 1, .1)
mydata4 <- rbinom(400, 1, .1)
## priors
alpha <- 2
beta <- 2
##
alpha2 <- 28
beta2 <- 4
##
```

Prior 1

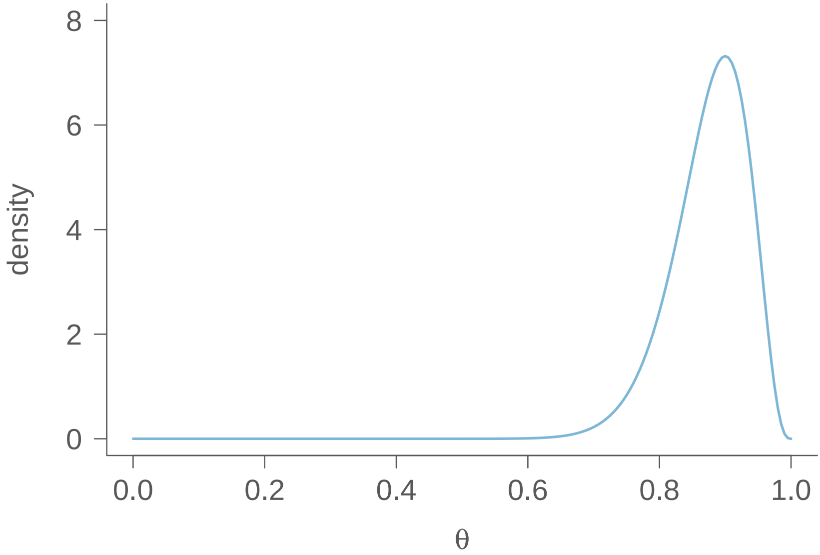


Data 1



prior
posterior 1
posterior 2
density
θ
0
2
4
6
8
10
12
0.0
0.2
0.4
0.6
0.8
1.0

Prior 2



Data 3-4 - prior 2