PASCAL AND THE INVENTION OF PROBABILITY THEORY

OYSTEIN ORE, Yale University*

Most textbooks on probability feel obliged to include a brief account of the history of the theory. Their descriptions of this process of initiation usually run somewhat in the following vein: "In the year 1654 a gambler named de Méré proposed to Pascal two problems which he had run across in his experiences at the gaming table."

It is likely that the distinguished Antoine Gombaud (or Gombauld), chevalier de Méré, sieur de Baussay, would turn in his grave at such a characterization of his main occupation in life. He certainly considered himself a model of courtly behavior and taught his esthetic principles elegantly to the *haut monde* as one may see from the frontispiece of his collected works. His writings appear today a little humorless and pedantic, but parts are still sufficiently entertaining to be readable, and they have secured him a permanent niche in the French literature of the seventeenth century.

De Méré (1607–1684) had received a good classical education and had served briefly in the army. His time was divided about equally between his small estate in Poitou and the court at Paris. His works show him to be a philosopher who in popular form expounded the ethics of a noble life, with particular emphasis upon the *agrêments* and the pleasant considerations for others which are essential for the *honnêtes hommes* of high society. He rapidly became a prominent figure at the court of Louis XIV where he was an adviser in delicate situations and an arbiter in conflicts. His charm, good taste, art in conversation, and correspondence made him an attractive guest in the salons and a friend of many of the important figures of his period. He recalled with particular pride his assistance to Madame de Maintenon before she became the favorite of the king. As time wore on he seems occasionally to have had rather exaggerated ideas of his own importance.

Sociability was his ideal; the criterion for good conversation was that it should be pleasant. Specialists were an abomination to him: "Most of them do not instruct at all since they rely upon obscure, sometimes even false, principles, and instead of seeking the truth to clarify it they aim to embarrass each other, also by terms which they do not even themselves understand, and by chimerical distinctions." It may even seem a little *mal à propos* for a mathematician to lecture on him, in view of de Méré's contempt for the scholars. He describes some categories of them as follows: "The best mathematicians who do not know how to entertain us except by numbers and figures, those knowing history by heart without ever having reflected upon it, or those who have a curious knowledge of many languages without having anything to say in them."

There is a little of this disdain in de Méré's description, in one of his letters, of how he made the acquaintance of Pascal: "I once made a trip with the duke

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of Roannez, who used to express himself with good and just sense and whom I found good company. Monsieur Mitton, whom you know and who is liked by all at court, was also with us, and because the trip was supposed to be a promenade rather than a voyage, we only thought of enjoying ourselves and we discussed everything. The duke was interested in mathematics, and in order to relieve tedium on the way he had provided a middle-aged man, who then was very little known, but who later certainly has made people talk about him. He was a great mathematician who knew nothing but that. These sciences give little sociable pleasure, and this man, who had neither taste nor sentiment, could not refrain from mingling into all we said, but he almost always surprised us and often made us laugh." De Méré goes on to tell that Pascal also carried strips of paper which he brought forth from time to time to write down some observations. After a few days Pascal came to enjoy the company and talked no more of mathematics.

This trip to Poitou probably took place in 1651 or 1652, but de Méré's account was written many years later and it seems that, in reminiscing, the chevalier's memory must have failed him to some extent. At the time, Blaise Pascal (1623–1662) was not yet thirty years old and could hardly be called a middle-aged man. Of course, he was almost constantly ill and may have aged prematurely. But he was already a well-known scientist. As a child prodigy he had accompanied his father to the meetings of the *Académie libre* in Paris, and when sixteen years old he had published a remarkable treatise on conic sections. At eighteen he had caused a great stir through the invention of his calculating machine, this *machine arithmétique* by means of which " . . . one could not only do all sorts of reckoning without feather or casters, but they could be done infallibly even if one did not know any of the rules of arithmetic." Pascal's demonstration of the weight of the atmosphere by barometric measurements at the base and summit of the Puy de Dôme peak had placed him in the forefront among contemporary physicists. Yet, in spite of all these achievements, Pascal's renown in Paris was still small in comparison with the fame he was later to acquire as the author of the *Lettres provinciales*. Perhaps, in retrospect, this was what the chevalier had in mind.

Pascal was at this time at the beginning of his so-called "worldly period," and, in his further account, de Méré even takes some credit for having brought it on. Certainly the two became well acquainted, and Pascal was influenced by de Méré's ideas on literary style. It has been indicated that de Méré advised Pascal on strategy in his attacks on the Jesuits in some of the *Lettres provinciales*.

The extent to which Pascal participated in the pleasures of life in the *haut monde* has been much argued by his biographers. His pious sisters feared that he was on the path to perdition. His interest in gaming and gambling questions have been cited as evidence of his dissipation in this period. Probably both Pascal and his friend de Méré spent some time at play; it was the fashionable pastime. But there is no indication that they did so with any passion; on the contrary, in their writings they both express themselves rather contemptuously against gambling. De Méré wrote to their common acquaintance Mitton about the pleasures of the countryside—two months is all he can stay in Paris before he gets homesick. "But I confess also that on my part I deplore you who are confined to gambling, longing for nothing but luck, without eyes for anything but this artificial world, almost like the courtesans to whom the great beauties of nature are unknown."

Pascal on his side tries to analyze the gambler's soul in the *Pensées*. "But you may say, what is his object? To boast tomorrow to his friends that he has played better than another? Such a man relieves the tedium of his life by playing every day. If you were to give him in the morning the money he might win during the day on condition that he should not gamble, you would make him unhappy. One may think possibly that he seeks the entertainment only of the game and not the gain. But let him play for nothing and he is bored and does not warm up to it. Therefore, it is not only the amusement which he seeks—languishing play without passion he finds tedious. He must become excited and deceive himself into believing that he would be happy to win that which he would not even accept were it not for the play. He must form an object for his passion to excite his desire, his anger, his fear, just like children who are scared of their own faces when they have blackened them."

Next let us turn to the two probability problems for which Pascal actually found solutions. It has often been stated that they were based upon de Méré's personal gambling experiences. This, as we shall see, seems very unlikely. While he was the first to call Pascal's attention to them, they were old and well-known questions. We shall take up first the so-called *dice problem*: When one throws with two dice, how many throws must one be allowed in order to have a better than even chance of getting two sixes at least once? Games of this kind were evidently popular in the middle ages; Cardano had dealt with them more than a century earlier, and there are still French dice games of a similar nature.

Let us recall briefly how the usual solution is obtained. It is convenient to determine first the probability of not obtaining any sixes. If one throws once there are thirty-six different possible throws with two dice and thirty-five of these do not give two sixes. Thus the probability of not getting two sixes in one throw is $q_1 = 35/36$. If one throws twice, there are 36×36 cases and 35×35 of them do not give two sixes either time. Thus $q_2 = (35/36)^2$. In the same way one finds that in n throws the probability of not getting any two sixes is $q_n = (35/36)^n$. Hence the opposite event, that of getting two sixes at least once, has the probability

$$p_n = 1 - q_n = 1 - \left(\frac{35}{36}\right)^n.$$

To have a better than even chance one must have $p_n > \frac{1}{2}$ and one finds

$$p_{24} = .4914, \qquad p_{25} = .5055.$$

Thus, only if one has 25 or more throws is it an advantageous proposition.

De Méré believed that the smallest advantageous number of throws should be 24. As the matter has been presented, he turned to Pascal because his own experiences had shown him that 25 throws were required. This is an unreasonable explanation. The difference between the probabilities for 24 and 25 throws is so small, as we have just seen, that to decide experimentally that one of them is less than $\frac{1}{2}$ would, according to modern statistical standards, require at least 100 sequences of trials, which in turn would involve several thousand individual throws with the two dice. Besides, the dice would have to be specially made in order to show no bias; the usual bone cubes turned out by the diciers of Paris would be much too inaccurate. To prepare special equipment of this kind and to keep the tedious records involved was evidently contrary to the chevalier's temperament.

However, Pascal's letters on probability to Pierre de Fermat (1601–1665), the learned jurist in Toulouse, throw light on the subject. In reply to an earlier letter from Fermat, Pascal writes, on July 29, 1654, "I admire much more your method for the division problem than that for the dice problem. I know that several persons have found the solution of the dice problem, as for instance, Monsieur le chevalier de Méré, who was the one who proposed these questions to me, and also M. de Roberval. But M. de Méré has never been able to determine the correct value in the division problem, nor the method to solve it, so that I found myself to be the only one who knew this proposition."

A little later in the same letter Pascal reports further on de Méré's views:

"He told me that the figures were wrong for the following reason: If one wants to throw a six with one die one has an advantage in four throws, as the odds are 671 to 625. If one shall throw two sixes with two dice there is a disadvantage in having only 24 throws. However, 24 to 36 (the number of cases for two dice) is as four to six (the number of cases on one die).

This was a great scandal which made him proclaim loudly that the theorems were not constant and Arithmetic belied herself. But you can easily see the reason for this result by the principles you possess."

Pascal does not understand de Méré's reasoning, and the passage also has been unintelligible to the biographers of Pascal. However, de Méré bases his objection upon an ancient gambling rule which Cardano also made use of: One wants to determine the *critical number* of throws, that is, the number of throws required to have an even chance for at least one success. If in one case there is one chance out of N_0 in a single trial, and in another one chance out of N_1, then the ratio of the corresponding critical numbers is as $N_0\colon N_1$. That is, we have

$$n_0 : N_0 = n_1 : N_1.$$

This immediately gives the proportion stated by de Méré. The rule was first proved by Abraham de Moivre (1667–1754) in his *Doctrine of Chances* (1716). If the chances are one in N_0 in a single trial, then the critical number is, with good approximation when N_0 is not too small,

$$n_0 = (\text{nat log } 2) \times N_0 = .69 \times N_0.$$

De Moivre applied this to the so-called Royal Oaks lottery in London; here there was one chance in 32, and so the critical number of trials by this rule is found to be $n_0 = 22.08$. The actual value is $n_0 = 22.135$, and, in general, de Moivre's rule gives very good values when N_0 is fairly large. However, de Méré made the error of believing that his gambling proportion was an absolute rule; for a small value, like $N_0 = 6$, the approximation to the actual value is not good enough.

These observations all indicate strongly that de Méré did not turn to Pascal with an actual gambling experience; rather, he was confused by the fact that a seemingly well-established gambling rule did not conform with the theoretical calculations which had been made. This tends to confirm the hypothesis that probability theory was at this time not in the state of absolute nonexistence that one is often led to believe. Cardano, around 1525, had already discovered certain rules which made it possible to solve the dice problem exactly, for one die. More than fifty years before Pascal, Galileo had given a complete table of probabilities for all throws with three dice. It appears likely that also in Pascal's circle in the mathematical academy, the simplest probability considerations were known. Pascal states that in addition to Fermat and himself, also de Méré and the mathematician Roberval could solve the dice problem.

In the preserved letters in the correspondence with Fermat, Pascal never refers to his own solution of the dice problem. Nor is it mentioned in Pascal's treatise on the *Arithmetic Triangle*, which was composed at this time and which includes the solution of the division problem as well as a few general probability principles. In regard to all other scientific achievements, Pascal always appeared anxious to receive proper recognition; in his pleasant correspondence with Fermat he insists on the importance of his own method in the division problem. In his investigations on the vacuum, he engaged in considerable argument to establish his priority, and the same is true in his later dispute regarding the cycloid or roulette curve. Thus there seems to be reason to believe that had Pascal had any feeling that this was an important discovery he would have expressed himself quite explicitly in regard to his priority rights.

The second problem which de Méré proposed was the *problème des parties*, commonly called the *division problem*. The question is how one shall divide equitably the prize money in a tournament in case the series for some reason is interrupted before it is completed. As we would say now, it is a question of determining the probability to win for each contestant, at a stage where each has a certain number of games or points to go. This is a problem of such difficulty that its solution by Pascal may well be considered a decisive break-through in the history of probability theory.

In this case it is still more evident that de Méré was not proposing a question from his own experience. For three centuries it had been a standard problem in mathematical texts. The first printed version may perhaps be found in Fra

Luca Paciuoli's *Summa* (1494), where, among the amusement questions, he proposes the following. "A team plays ball such that a total of 60 points is required to win the game, and each inning counts 10 points. The stakes are 10 ducats. By some incident they cannot finish the game and one side has 50 points and the other 20. One wants to know what share of the prize money belongs to each side. In this case I have found that opinions differ from one to another, but all seem to me insufficient in their arguments, but I shall state the truth and give the correct way."

His second example is a shooting match. "Three compete with the cross bow and the one who first obtains six first places wins; they stake 10 ducats among themselves. When the first has four best hits, the second three, and the third two, they do not want to continue and decide to divide the prize fairly. One asks what the share of each should be." Fra Luca also warns against gamblers who play "morra" to five points for five ducats and when they are behind, for instance four wins to three, " . . . they say, we come back . . . ," and want the prize divided three to two. Otherwise there is a noticeable avoidance of formulating the problem for a straight gambling game, presumably to avoid criticism for dealing with objectionable pastimes.

Most accounts of the division problem take their starting point in Fra Luca's examples. However, the problem is a much older one. I have found it in Italian mathematical manuscripts as early as 1380. It seems likely that it is of Arabic origin. It does not appear in Leonardo Fibonacci's *Liber abaci* (1202), which brought many Arabic puzzles to Italy, but the form of the problem is reminiscent of the distribution and inheritance problems of the Arabs.

The Renaissance mathematicians made only trivial contributions to the division problem, although those who deal with it make great claims for their own methods and are liberal in their criticisms of others. Cardano, for instance, says about the solution by Paciuoli, "And there is an evident error in the determination of the shares in the game problem as even a child should recognize, while he (Paciuoli) criticizes others and praises his own excellent opinion."

Cardano's arch enemy, Tartaglia, feels himself on swaying ground when he deals with the division problem in his *General Trattato* (1556). The margin displays the warning, "*Error di Fra Luca dal Borgo*," and Tartaglia gives his own rule, but with the reservation: "Therefore I say that the resolution of such a question is judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation." Toward the end of the chapter, Tartaglia considers these matters as having "*poco sugo*" and giving rise "to great dispute and so it appears to me better not to speak more of this matter, although some people like such facetious questions in order to have an occasion to create an argument."

The division problem remained in the arithmetic texts until well into the seventeenth century. It can be found also in French books, and de Méré may have read it in his own school. The form changed somewhat with time; let me cite a couple of examples from an arithmetic by Forestani (Venice 1603). "An elderly nobleman, staying at his country house, was extremely fond of watching ball games, and so he called in two young farmhands, saying, 'Here are four ducats for which you may play; the one who first takes eight games is the winner.' So they began to play, but when one had five games and the other three games, they lost the ball and were unable to finish. The question is how the prize should be divided."

Another runs as follows: "Three soldiers garrisoned at a fortress take a walk and find a scudo. Each of them claims it; however at last they agree that they will play a match of pallatella to fourteen games, and he who wins shall have the scudo." When they have won respectively ten, eight, and five games, they are called to guard duty and the proper shares shall again be determined.

Pascal probably found the solution of the division problem early in the spring of the year 1654. The method depends on binomial coefficients, and Pascal was undoubtedly greatly aided by his studies of the *Arithmetical Triangle*, a table of such coefficients. The solution was based upon a somewhat artificial approach. When the two players needed a and b games respectively to win, Pascal lets them play altogether $a+b-1$ games, regardless of whether the series might have been decided before this many games. The argument is quite correct and is still in common use in textbooks on probability. However, the mathematician Roberval objected strenuously, and a discussion with Roberval was usually not pleasant. One of his contemporaries called him "the greatest mathematician in Paris, and in conversation the most disagreeable man in the world."

These criticisms by Roberval seem to have been the immediate reason why Pascal sought the opinion of Fermat, the recognized grand master of mathematics in France at the time. Carcavy, the royal librarian, also a member of the scientific circle in Paris, acted as an intermediary; he had formerly been Fermat's colleague as a judge at the parliament court in Toulouse.

It will carry us too far to give a detailed account of this correspondence, which lasted through the summer and fall of 1654. Fermat was delighted to come into closer contact with the young Pascal; he had previously been on friendly terms with his father. Fermat had begun to feel the scientific isolation in Toulouse and hoped that Pascal might assist him in publishing his mathematical results. In one of his letters to Carcavy he writes, "I have been delighted to have my own sentiments conform to those of M. Pascal, for I admire his genius infinitely and believe he is capable of achieving anything he may undertake. The friendship which he offers to me is so precious and so considerable that I believe it will not be disturbed if I should make some use of it in the printing of my treatises."

Pascal, on the other hand, was cheered by finding that Fermat's results were in complete agreement with his own: "I see that the truth is the same in Toulouse and Paris." Pascal at this time was absorbed in an intense scientific production; among other things he completed his *Traité du Triangle Arithmétique* with an extensive discussion of the division problem. After his death this work was found fully printed, but unpublished, among his posthumous papers. The reason is well known. On November 23, 1654, in Paris, the crisis occurred which has been called Pascal's definitive conversion. From now on his life and thoughts revolved almost exclusively around the questions of his Christian faith. He joined his sister as a member of the Jansenist group for whom scientific studies were but illusions serving to distract from the final purpose of life, the salvation of the soul.

But Pascal never quite relinquished his interest in the newly created field of probability. In his famous *Pensées*, one of the most curious sections is "*Le pari*," the wager, a dialogue about the existence of God. It is at first difficult to understand and it has been widely discussed. But if one recognizes that Pascal has a definite mathematical probability formula in mind, the passage becomes quite lucid. It has been conjectured that Pascal conceived of "*Le pari*" as a dialogue between himself and his unbelieving friend de Méré:

> Pascal: God exists or he does not. Which side shall we take. Reason can decide nothing. An infinite chaos separates us. A game is being played where a decision, heads or tails, will be made at the end of this infinite distance. On what do you place your bet? By reason you cannot take one or the other; by reason you can defend neither choice. Therefore, do not blame the error of those who have made a choice, since you know nothing about it.
>
> De Méré: No, but I blame them for having made a choice at all, not for their particular choice, for they are equally at fault, both he who chooses heads and he who chooses tails. The correct attitude is not to bet at all.
>
> Pascal: Yes, but one is compelled to wager, it is not voluntary, you are in the game. Which side do you take? Let us see. Since you must wager, let us find out which alternative is the least profitable.

Pascal goes on to argue, on the principle of mathematical expectation, that the value of a game is the prize to be won times the probability for winning it. This should be compared to the amount risked times the probability for losing.

> Let us see: since there is an equal chance of gain or loss, and if you were to win only two lives for one, you should still bet. But if there were three to win you would also play—since you are compelled to—and you would be imprudent not to risk your life to win three others in a game with such chances to win or lose. But there is an eternity of life and happiness. And when this is so, if there should be an infinite number of chances with only a single favorable to you, it would still be right to bet one to obtain two; you would act with bad judgment if, when obliged to play, you would refuse to stake one life against three, even if there is an infinity of chances and but one for you, provided there is an infinite life of infinite happiness to be gained. But here, actually, there is such an infinite life of infinite happiness to be won, one chance of winning against a finite number of possibilities for a loss, and that which you risk is finite. This eliminates all choice; whenever an infinite gain is involved and there is not an infinite number of losing chances against the winning ones, there is nothing to weigh, one must give all. And so, when forced to play, one must sacrifice reason to win life rather than to stake it against the infinite profit which may accrue just as easily as the loss: annihilation.

Pascal continues in the same vein, and in the end de Méré seems to be convinced, but finally he asks somewhat irreverently, "I confess, this I admit, but then is there no way of looking at the underside of the cards which have been dealt?" To this Pascal replies, "Yes, the Holy Scriptures and the rest."

Much criticism has been expressed against Pascal's logic, and one must admit that this is hardly a field for applied mathematics. But even Pascal himself cannot have believed that his argument should carry a convincing mathematical weight; however, if his design was to create a thought-provoking parable, he succeeded admirably.

It was well known in Paris that Pascal and Fermat had disccovered a new branch of mathematics, but no one knew much about the details. The young Dutch genius Christiaan Huygens arrived in Paris less than a year later. He had worked on a couple of problems concerning games of chance and was anxious to consult with one of the principals in probability theory. Huygens was well received in Paris. He was introduced to the members of the informal academy, and made the acquaintance of Roberval and the lawyer and amateur mathematician Mylon. But Fermat was far away in Toulouse, and after his conversion Pascal admitted no visitors. Huygens did not receive the information he desired, but after his return to Holland he began drafting his own little treatise on probability, *Calculations in Games of Chance*. But he was concerned about the correctness of his own results, and to obtain a check on them he sent one of his problems to Roberval, Mylon, and Carcavy. Mylon replied with an erroneous answer which Huygens politely corrected, and from Roberval we have no report. Carcavy, however, consulted with his close friend Pascal, and also forwarded the problem to Fermat. Fermat promptly confirmed Huygens' solution and included for Huygens' consideration a series of five problems, which are reproduced at the end of the little treatise.

Encouraged by this contact, Huygens wrote another letter to Carcavy and a little later, much to his surprise, he received a letter from Mylon stating that Pascal had found his principle admirable and in conformity with his own procedure. Mylon explained that "although it is very difficult to meet Pascal since he has retired completely to give himself entirely to devotion, he has not lost his mathematics from view. When M. de Carcavy can visit him and propose some problem to him, he does not refuse to give the solution, particularly in the field of games of chance which he was the first to bring under discussion. Since I do not possess the same goodness as these gentlemen, I have all the difficulties in the world to meet them, since they are entirely absorbed in religious affairs and I only rarely visit those places."

Equally surprising was the fact that Pascal had given Carcavy a gambling problem to transmit. "*A* and *B* play at hazard with three dice and fixed points fourteen and eleven respectively. Each has twelve pennies and receives one penny from the other every time his own point turns up. What are the odds for one player to ruin the other?"

This problem Huygens included as the last in his collection of exercises for the readers of his booklet on probability. It is far more difficult than the rest, and it embodies, in spite of its innocuous form, the beginnings of a whole field of probability, the theory of random walks, Brownian motion, and other questions from the kinetic gas theory.

Pascal's moral views evidently did not permit him to propose a plain gambling proposition, so that in his original version as reported by Carcavy he only lets the players make pencil marks on paper. Pascal was keenly aware of the difficulties involved and he had the hope that he had invented a problem which might stymie even the formidable Fermat. In the correspondence between the two, one has a feeling that Pascal is a little dismayed at the apparent ease with which the old master tackled his problems. But also this time Fermat immediately returned a solution which agreed with the one found by Pascal.

Huygens was elated at the confirmation of his methods, but regretted deeply his failure to meet Pascal: "If one had not assured me while I was in Paris that he had abandoned the study of mathematics entirely I should have tried by every means to make his acquaintance." Later Huygens corresponded with Pascal on other matters, and when Huygens returned to Paris in 1660, a couple of years before Pascal's death, the two met on several occasions.

Huygens also became friendly with the duke of Roannez and in his diary he relates that he was once entertained at dinner in the ducal palace in the company of the chevalier de Méré, "inventor of the division in games." It is noteworthy that in all the discussion about probability problems no one thought it worth while to consult with de Méré. Pascal, in one of his letters to Fermat, made the comment upon him, "He is a good wit, but not a mathematician."

Nevertheless, the fact that de Méré had been a figure, albeit a minor one, in the creation of a new mathematical field seems to have gone to his head. To the consternation of contemporary scientists he wrote a letter to Pascal in the following vein.

> "Do you remember you once told me that you were no longer convinced of the excellence of mathematics? You write to me this time that I have disillusioned you completely and also that I have discovered things which you would never have perceived if you had not known me. I don't know, however, Monsieur, if you are as obliged to me as you may think. You still have the habit, which you have gathered from this science, not to judge anything except from your demonstrations, which are often false. These long reasonings drawn from line to line prevent you from obtaining the higher point of view which never deceives."

Later on he admonishes Pascal as follows:

> "You know that I have discovered such rare things in mathematics that the most learned among the ancients have never discussed them and they have surprised the best mathematicians in Europe. You have written on my inventions, as well as Monsieur Huygens, Monsieur de Fermat, and many others who have admired them. You may conclude from this that I do not propose to anyone to scorn this science and truly, it may be of service provided one does not attach oneself too closely to it, for ordinarily, that which one seeks with so much curiosity appears useless to me and the time spent at it could be better employed."

De Méré's outburst seems to have amused the court and a wit proposed epigrammatically that the chevalier believed "he could teach Madame de Maintenon courtly behavior and Pascal mathematics."

Peculiarly enough, the whole letter was printed verbatim in Boyle's important encyclopedia *Dictionaire historique et critique.* When Leibniz came across the article he commented:

"I almost laughed at the airs which the chevalier de Méré takes on in his letter to Pascal which Boyle reports in the same article. But I notice that the chevalier knew that Pascal's genius also had its weak sides which sometimes made him susceptible to the influence of too extravagant spiritualists and even at times made him lose the taste for solid knowledge.

M. de Méré takes advantage of this to talk down to Pascal. It seems to me that he makes a little fun of him, as men of the world often do when they have an abundance of esprit, but mediocre knowledge. They want to convince us that those things which they do not sufficiently understand are but of small value; one should send them to school with Roberval. It is true, nevertheless, that the chevalier was unusually gifted even in mathematics."

Leibniz goes on with a brief mention of some of the men who had worked with probability problems. He himself often showed an interest in the subject but never made any contributions of consequence. However, he concludes his epistle with the judgment which has never been more true than at present, "So also the games in themselves merit to be studied and if some penetrating mathematician meditated upon them he would find many important results, for man has never shown more ingenuity than in his plays."

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INTEGRATION BY PARTS FOR STIELTJES INTEGRALS

EDWIN HEWITT, University of Washington*

There are several formulas for integration by parts. The most familiar of these, as found in the usual calculus text, states that

$$\int_a^b f(t)g'(t)dt + \int_a^b f'(t)g(t)dt = f(b)g(b) - f(a)g(a). \tag{1}$$

Clearly (1) does not hold without some restriction on the functions f and g. (We shall point out below the class of functions to which it is appropriate to apply (1).)

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